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Dynamic analysis of a multi-span uniform beam carrying a number of various concentrated elements

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Abstract

This paper employs the numerical assembly method (NAM) to determine the "exact" frequency–response amplitudes of a multiple-span beam carrying a number of various concentrated elements and subjected to a harmonic force, and the exact natural frequencies and mode shapes of the beam for the case of zero harmonic force. First, the coefficient matrices for the intermediate concentrated elements, pinned support, applied force, left-end support and right-end support of a beam are derived. Next, the overall coefficient matrix for the whole vibrating system is obtained using the numerical assembly technique of the conventional finite element method (FEM). Finally, the exact dynamic response amplitude of the forced vibrating system corresponding to each specified exciting frequency of the harmonic force is determined by solving the simultaneous equations associated with the last overall coefficient matrix. The graph of dynamic response amplitudes versus various exciting frequencies gives the frequency–response curve for any point of a multiple-span beam carrying a number of various concentrated elements. For the case of zero harmonic force, the above-mentioned simultaneous equations reduce to an eigenvalue problem so that natural frequencies and mode shapes of the beam can also be obtained.

1. Introduction

A lot of reports have been published in the area about the vibration characteristics of a uniform beam carrying various concentrated elements (such as point masses, rotary inertias, linear springs, rotational springs, spring-mass systems, etc.). In Refs. [1–4], various techniques were presented to perform the forced vibration analysis of beams carrying one or two concentrated elements. Wu et al. found natural frequencies and mode shapes of a uniform beam carrying any number of rigidly attached point masses [5] and elastically attached point masses [6] by means of the analytical-and-numerical combined method. Naguleswaran [7] found the natural frequencies of an Euler–Bernoulli beam with up to five elastic supports (including ends of beam) by setting a fourth-order determinant to be zero. Wu and Chou [8] obtained the exact solution of a uniform beam carrying any number of spring-mass systems by using the numerical assembly method (NAM). Chen [9] studied the free vibration problem concerning uniform and non-uniform "single-span" beams

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Nomenclature

$\bar{F}$	amplitude of external force
F_v	harmonic force at the vth station
E	Young's modulus
I	second moment of cross-sectional area of the beam
i	ith beam segment
J_u	rotary inertia at the uth station
j	$\sqrt{-1}$
K_{Ru}	rotational spring constant at the uth station
K_{Tu}	translational (linear) spring constant at the uth station
L	total length of the beam
$\bar{m}$	mass per unit length of the beam
m_u	lumped mass at the uth station
n	total number of intermediate stations
q	total number of equations for the integration constants
x_u	coordinate of station u
$y(x,t)$	transverse displacement at position x and time t for the beam
$Y(x)$	amplitude of $y(x,t)$
$\bar{Y}(x)$	dimensionless amplitude of $y(x,t)$
Ω	dimensionless frequency parameter for the beam
ω	natural frequency of the beam
ω_e	exciting frequency of the applied harmonic force
ξ_t	dimensionless coordinate of tth station ($= x_t/L$)

carrying various concentrated elements. Employing the same technique as Chen [9], Lin and Tsai determined the exact values of natural frequencies and associated mode shapes of a "multi-span" beam carrying a number of point masses, spring-mass systems [10,11] and "multi-step" beam carrying a number of point masses and rotary inertias [12]. Gürgöze and Erol [13,14] studied the forced vibration responses of a cantilever beam with a "single" intermediate support.

The objective of this paper is to adopt the NAM to investigate the free and forced vibration characteristics of a "multiple-span" uniform beam carrying a number of various concentrated elements and subjected to a harmonic force.

2. Equation of motion and displacement function

Fig. 1 shows the sketch of a uniform beam with pinned–pinned (P–P) boundary conditions and R intermediate pinned supports. It carries U various concentrated elements (including point masses, rotary inertias, linear springs and/or rotational springs) and subjected to V harmonic forces. The points

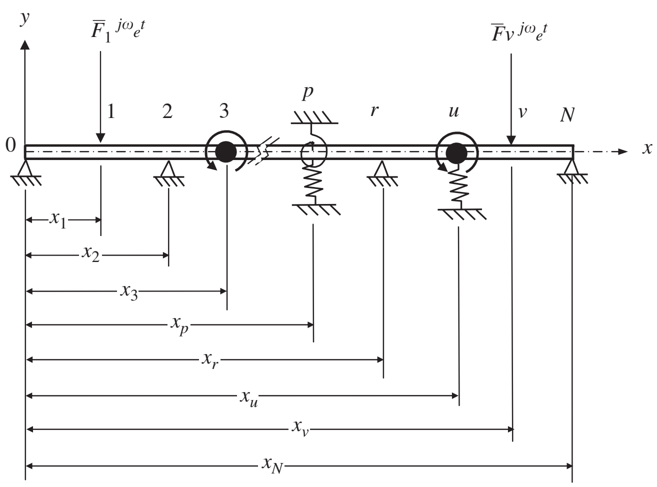

Fig. 1. Sketch for a uniform beam supported by pins, carrying various concentrated elements and subjected to harmonic concentrated forces.

corresponding to the locations of pinned supports, applied concentrated forces and various concentrated elements are called "stations" in this paper.

The differential equation of motion for a uniform beam (cf. Fig. 1) with small deflections is given by

$$EI\frac{\partial^4 y(x,t)}{\partial x^4} + \bar{m}\frac{\partial^2 y(x,t)}{\partial^2 t} = F(t)\delta(x - x_i), \tag{1}$$

where E is Young's modulus, I is second moment of the cross-sectional area, $\bar{m}$ is mass per unit length of the beam, $y(x,t)$ is transverse displacement at position x and time t and $F(t)$ is a force (with its magnitude equal to external load per unit length) at time t. Besides, $\delta(x-x_i)$ is the Dirac delta with x_i denoting the coordinate at which the concentrated force $F(t)$ applied.

If the applied concentrated force takes the form

$$F(t) = \bar{F}e^{j\omega_e t}, \tag{2}$$

then, in the steady state, one has

$$y(x,t) = Y(x)\,e^{j\omega_e t}, \tag{3}$$

where $Y(x)$ is the amplitude of $y(x,t)$, ω_e is the exciting frequency of the applied harmonic forces, $\bar{F}$ is amplitude of $F(t)$ and $j = \sqrt{-1}$.

Substitution of Eqs. (2) and (3) into Eq. (1) gives

$$Y'''' - \beta^4 Y = \frac{\bar{F}}{EI}\delta(x - x_i), \tag{4}$$

where

$$\beta^4 = \frac{\omega_e^2 \bar{m}}{EI} \tag{5a}$$

or

$$\omega_e = (\Omega)^2\left(\frac{EI}{\bar{m}L^4}\right)^{1/2} \tag{5b}$$

with

$$\Omega = \beta L = \left(\omega_e^2\frac{\bar{m}L^4}{EI}\right)^{1/4}. \tag{5c}$$

Eq. (4) is a non-homogeneous equation, its "complete" solution takes the form

$$Y(x) = (C_1 \sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x) - \frac{\bar{F}}{\beta^4 EI}\delta(x - x_i), \tag{6a}$$

in which C_1, C_2, C_3 and C_4 are the unknown integration constants. From Eq. (6a) one sees that, for any beam segment on which no concentrated force $F(t)$ (with amplitude $\bar{F}$) being applied, its displacement function will take the form

$$Y(x) = (C_1 \sin\beta x + C_2\cos\beta x + C_3\sinh\beta x + C_4\cosh\beta x). \tag{6b}$$

In other words, only the beam segment with concentrated force $F(t)$ being applied at its end (or node), Eq. (6a) should be used to derive the equilibrium equations concerned.

3. Coefficient matrices for intermediate stations and ends of the beam

For an arbitrary point located at x_t (cf. Fig. 1), one obtains from Eq. (6a) or (6b)

$$Y_t'(\xi_t) = \Omega C_{t,1}\cos\Omega\xi_t - \Omega C_{t,2}\sin\Omega\xi_t + \Omega C_{t,3}\cosh\Omega\xi_t + \Omega C_{t,4}\sinh\Omega\xi_t, \tag{7a}$$

$$Y_t''(\xi_t) = -\Omega^2 C_{t,1}\sin\Omega\xi_t - \Omega^2 C_{t,2}\cos\Omega\xi_t + \Omega^2 C_{t,3}\sinh\Omega\xi_t + \Omega^2 C_{t,4}\cosh\Omega\xi_t, \tag{7b}$$

$$Y'''_t(\xi_t) = -\Omega^3 C_{t,1} \cos \Omega\xi_t + \Omega^3 C_{t,2} \sin \Omega\xi_t + \Omega^3 C_{t,3} \cosh \Omega\xi_t + \Omega^3 C_{t,4} \sinh \Omega\xi_t, \tag{7c}$$

where

$$\xi_t = \frac{x_t}{L}. \tag{8}$$

3.1. Coefficient matrix $[B_u]$ for an intermediate concentrated element

If the station numbering of an intermediate concentrated element including point mass, rotary inertia, linear spring or rotational spring is u, then the continuity of deformations and equilibrium of the moments and forces require that

$$Y^L_u(\xi_u) = Y^R_u(\xi_u), \tag{9a}$$

$$Y'^L_u(\xi_u) = Y'^R_u(\xi_u), \tag{9b}$$

$$Y''^L_u(\xi_u) + (K^*_{Ru} - \Omega^4 J^*_u) Y'^L_u(\xi_u) = Y''^R_u(\xi_u), \tag{9c}$$

$$Y'''^L_u(\xi_u) + (\Omega^4 m^*_u - K^*_{Tu}) Y^L_u(\xi_u) = Y'''^R_u(\xi_u), \tag{9d}$$

where

$$m^*_u = \frac{m_u}{\bar{m}L}, \quad J^*_u = \frac{J_u}{\bar{m}L^3}, \quad K^*_{Tu} = \frac{K_{Tu}L^3}{EI}, \quad K^*_{Ru} = \frac{K_{Ru}L}{EI}. \tag{10a,b,c,d}$$

In Eqs. (10a–d), m_u, J_u, K_{Tu} and K_{Ru} are respectively the lumped mass, rotary inertia, linear spring constant and rotational spring constant at the uth station, while the right superscripts L and R in Eqs. (9a)–(9d) refer to the ''left'' and ''right'' sides of station u.

Substitution of Eqs. (6b), (7a)–(7c) into Eqs. (9a)–(9d) leads to

$$\begin{aligned} &C_{u,1} \sin \Omega\xi_u + C_{u,2} \cos \Omega\xi_u + C_{u,3} \sinh \Omega\xi_u + C_{u,4} \cosh \Omega\xi_u - C_{u+1,1} \sin \Omega\xi_u \\ &\quad - C_{u+1,2} \cos \Omega\xi_p - C_{u+1,3} \sinh \Omega\xi_u - C_{u+1,4} \cosh \Omega\xi_u = 0, \end{aligned} \tag{11a}$$

$$\begin{aligned} &C_{u,1} \cos \Omega\xi_u - C_{u,2} \sin \Omega\xi_u + C_{u,3} \cosh \Omega\xi_u + C_{u,4} \sinh \Omega\xi_u - C_{u+1,1} \cos \Omega\xi_u \\ &\quad + C_{u+1,2} \sin \Omega\xi_p - C_{u+1,3} \cosh \Omega\xi_u - C_{u+1,4} \sinh \Omega\xi_u = 0, \end{aligned} \tag{11b}$$

$$\begin{aligned} &\Omega(-C_{u,1} \sin \Omega\xi_u - C_{u,2} \cos \Omega\xi_u + C_{u,3} \sinh \Omega\xi_u + C_{u,4} \cosh \Omega\xi_u) \\ &\quad - (\Omega^4 J^*_u - K^*_{Ru})(C_{u,1} \cos \Omega\xi_u - C_{u,2} \sin \Omega\xi_u + C_{u,3} \cosh \Omega\xi_u + C_{u,4} \sinh \Omega\xi_u) \\ &\quad + \Omega(C_{u+1,1} \sin \Omega\xi_u + C_{u+1,2} \cos \Omega\xi_u - C_{u+1,3} \sinh \Omega\xi_u - C_{u+1,4} \cosh \Omega\xi_u) = 0, \end{aligned} \tag{11c}$$

$$\begin{aligned} &\Omega^3(-C_{u,1} \cos \Omega\xi_u + C_{u,2} \sin \Omega\xi_u + C_{u,3} \cosh \Omega\xi_u + C_{u,4} \sinh \Omega\xi_u) \\ &\quad + (\Omega^4 m^*_u - K^*_{Tu})(C_{u,1} \sin \Omega\xi_u + C_{u,2} \cos \Omega\xi_u + C_{u,3} \sinh \Omega\xi_u + C_{u,4} \cosh \Omega\xi_u) \\ &\quad + \Omega^3(C_{u+1,1} \cos \Omega\xi_u - C_{u+1,2} \sin \Omega\xi_u - C_{u+1,3} \cosh \Omega\xi_u - C_{u+1,4} \sinh \Omega\xi_u) = 0, \end{aligned} \tag{11d}$$

Writing Eqs. (11a)–(11d) in matrix form, one has

$$[B_u]\{C_u\} = 0, \tag{12}$$

where

$$\{C_u\} = \left\{ C_{u,1} \quad C_{u,2} \quad C_{u,3} \quad C_{u,4} \quad C_{u+1,1} \quad C_{u+1,2} \quad C_{u+1,3} \quad C_{u+1,4} \right\} \tag{13}$$

and the coefficient matrix $[B_u]$ is placed in Eq. (A.1) of the appendix at the end of this paper. In the above expressions, the symbols, [ ] and { }, denote the rectangular matrix and column vector, respectively.

3.2. Coefficient matrix $[B_r]$ for an intermediate rigid support

Similarly, if the station numbering of an intermediate rigid support is r, then the continuity of deformations and equilibrium of moments require that

$$Y_r(\xi_r) = Y_{r+1}(\xi_r) = 0, \tag{14a,b}$$

$$Y'_r(\xi_r) = Y'_{r+1}(\xi_r), \tag{14c}$$

$$Y''_r(\xi_r) = Y''_{r+1}(\xi_r). \tag{14d}$$

Introducing Eqs. (6b), (7a) and (7b) into Eqs. (14), one obtains

$$C_{r,1} \sin \Omega\xi_r + C_{r,2} \cos \Omega\xi_r + C_{r,3} \sinh \Omega\xi_r + C_{r,4} \cosh \Omega\xi_r = 0, \tag{15a}$$

$$C_{r+1,1} \sin \Omega\xi_r + C_{r+1,2} \cos \Omega\xi_r + C_{r+1,3} \sinh \Omega\xi_r + C_{r+1,4} \cosh \Omega\xi_r = 0, \tag{15b}$$

$$\begin{aligned} &C_{r,1} \cos \Omega\xi_r - C_{r,2} \sin \Omega\xi_r + C_{r,3} \cosh \Omega\xi_r + C_{r,4} \sinh \Omega\xi_r \\ &\quad - C_{r+1,1} \cos \Omega\xi_r + C_{r+1,2} \sin \Omega\xi_r - C_{r+1,3} \cosh \Omega\xi_r - C_{r+1,4} \sinh \Omega\xi_r = 0, \end{aligned} \tag{15c}$$

$$\begin{aligned} &- C_{r,1} \sin \Omega\xi_r - C_{r,2} \cos \Omega\xi_r + C_{r,3} \sinh \Omega\xi_r + C_{r,4} \cosh \Omega\xi_r \\ &+ C_{r+1,1} \sin \Omega\xi_r + C_{r+1,2} \cos \Omega\xi_r - C_{r+1,3} \sinh \Omega\xi_r - C_{r+1,4} \cosh \Omega\xi_r = 0 \end{aligned} \tag{15d}$$

or

$$[B_r]\{C_r\} = 0 \tag{16}$$

where

$$\{C_r\} = \{\, C_{r,1} \quad C_{r,2} \quad C_{r,3} \quad C_{r,4} \quad C_{r+1,1} \quad C_{r+1,2} \quad C_{r+1,3} \quad C_{r+1,4} \,\}. \tag{17}$$

The coefficient matrix $[B_r]$ is placed in Eq. (A.2) of the appendix.

3.3. Coefficient matrix $[B_v]$ for an intermediate applied force

If the station numbering for the intermediate harmonic concentrated force normal to the beam is v, then the continuity of deformations and equilibrium of moments and forces require that

$$Y^L_v(\xi_v) = Y^R_v(\xi_v), \tag{18a}$$

$$Y'^L_v(\xi_v) = Y'^R_v(\xi_v), \tag{18b}$$

$$Y''^L_v(\xi_v) = Y''^R_v(\xi_v), \tag{18c}$$

$$Y'''^L_v(\xi_v) + \frac{\bar{F}_v}{EI} = Y'''^R_v(\xi_v). \tag{18d}$$

Introducing Eqs. (6a), (7a) and (7c) into Eqs. (18a) and (18d), one obtains

$$\begin{aligned} &C_{v,1} \sin \Omega\xi_v + C_{v,2} \cos \Omega\xi_v + C_{v,3} \sinh \Omega\xi_v + C_{v,4} \cosh \Omega\xi_v \\ &- C_{v+1,1} \sin \Omega\xi_v - C_{v+1,2} \cos \Omega\xi_v - C_{v+1,3} \sinh \Omega\xi_v - C_{v+1,4} \cosh \Omega\xi_v = 0, \end{aligned} \tag{19a}$$

$$\begin{aligned} &C_{v,1} \cos \Omega\xi_v - C_{v,2} \sin \Omega\xi_v + C_{v,3} \cosh \Omega\xi_v + C_{v,4} \sinh \Omega\xi_v \\ &- C_{v+1,1} \cos \Omega\xi_v + C_{v+1,2} \sin \Omega\xi_v - C_{v+1,3} \cosh \Omega\xi_v - C_{v+1,4} \sinh \Omega\xi_v = 0, \end{aligned} \tag{19b}$$

$$\begin{aligned} &- C_{v,1} \sin \Omega\xi_v - C_{v,2} \cos \Omega\xi_v + C_{v,3} \sinh \Omega\xi_v + C_{v,4} \cosh \Omega\xi_v \\ &+ C_{v+1,1} \sin \Omega\xi_v + C_{v+1,2} \cos \Omega\xi_v - C_{v+1,3} \sinh \Omega\xi_v - C_{v+1,4} \cosh \Omega\xi_v = 0, \end{aligned} \tag{19c}$$

$$\Omega^3(-C_{v,1}\cos\Omega\xi_v + C_{v,2}\sin\Omega\xi_v + C_{v,3}\cosh\Omega\xi_v + C_{v,4}\sinh\Omega\xi_v)$$
$$+\,\Omega^3(C_{v+1,1}\cos\Omega\xi_v - C_{v+1,2}\sin\Omega\xi_v - C_{v+1,3}\cosh\Omega\xi_v - C_{v+1,4}\sinh\Omega\xi_v) = -\frac{\bar{F}_v L^3}{EI}. \tag{19d}$$

Writing Eqs. (19a)–(19d) in matrix form, one has

$$[B_v]\{C_v\} = \{D_v\}. \tag{20}$$

The coefficient matrix $[B_v]$ is placed in Eq. (A.3) of the appendix, and

$$\{C_v\} = \{\, C_{v,1} \quad C_{v,2} \quad C_{v,3} \quad C_{v,4} \quad C_{v+1,1} \quad C_{v+1,2} \quad C_{v+1,3} \quad C_{v+1,4} \,\}, \tag{21}$$

$$\{D_v\} = \left\{ 0 \quad 0 \quad 0 \quad -\frac{\bar{F}_v L^3}{EI} \right\}. \tag{22}$$

3.4. Coefficient matrix $[B_0]$ for left end of the entire beam

If the left-end support of the beam is pinned as shown in Fig. 1, then the boundary conditions are

$$Y_0(0) = Y_0''(0) = 0. \tag{23a,b}$$

The substitution of Eqs. (6b) and (7b) into Eqs. (23a) and (23b) leads to

$$C_{0,2} + C_{0,4} = 0, \tag{24a}$$

$$-C_{0,2} + C_{0,4} = 0 \tag{24b}$$

or in matrix form

$$[B_0]\{C_0\} = 0, \tag{25}$$

where

$$[B_0] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}, \tag{26}$$

$$\{C_0\} = \{\, C_{0,1} \quad C_{0,2} \quad C_{0,3} \quad C_{0,4} \,\}. \tag{27}$$

If the left-end support of the beam is free, then the boundary conditions are

$$Y_0''(0) = Y_0'''(0) = 0. \tag{28a,b}$$

From Eqs. (7b), (7c) and (28), one obtains the following boundary coefficient matrix:

$$[B_0] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}. \tag{29}$$

If the left-end support of the beam is clamped, one obtains the following boundary coefficient matrix:

$$[B_0] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}. \tag{30}$$

3.5. Coefficient matrix $[B_N]$ for right end of the entire beam

If the right-end support of the beam is pinned as shown in Fig. 1, then the boundary conditions are

$$Y_N(L) = Y''_N(L) = 0 \tag{31a,b}$$

with

$$N = n + 1, \tag{32}$$

where n is total number of stations.

Substituting Eqs. (6b) and (7b) into Eqs. (31a) and (31b) give

$$C_{N,1} \sin \Omega + C_{N,2} \cos \Omega + C_{N,3} \sinh \Omega + C_{N,4} \cosh \Omega = 0, \tag{33a}$$

$$-C_{N,1} \sin \Omega - C_{N,2} \cos \Omega + C_{N,3} \sinh \Omega + C_{N,4} \cosh \Omega = 0 \tag{33b}$$

or

$$[B_N]\{C_N\} = 0, \tag{34}$$

where

$$[B_N] = \begin{array}{c} \begin{array}{cccc} 4N-3 & 4N-2 & 4N-1 & 4N \end{array} \\ \begin{bmatrix} \sin \Omega & \cos \Omega & \sinh \Omega & \cosh \Omega \\ -\sin \Omega & -\cos \Omega & \sinh \Omega & \cosh \Omega \end{bmatrix} \end{array} \begin{array}{c} \\ q-1 \\ q \end{array}, \tag{35}$$

$$\{C_N\} = \{\, C_{N,1} \quad C_{N,2} \quad C_{N,3} \quad C_{N,4} \,\}. \tag{36}$$

If the right-end support of the beam is clamped, then the boundary conditions are

$$Y_N(L) = Y'_N(L) = 0. \tag{37a,b}$$

From Eqs. (6b), (7a) and (37), one obtains the following boundary coefficient matrix:

$$[B_N] = \begin{array}{c} \begin{array}{cccc} 4N-3 & 4N-2 & 4N-1 & 4N \end{array} \\ \begin{bmatrix} \sin \Omega & \cos \Omega & \sinh \Omega & \cosh \Omega \\ \cos \Omega & -\sin \Omega & \cosh \Omega & \sinh \Omega \end{bmatrix} \end{array} \begin{array}{c} \\ q-1 \\ q \end{array}, \tag{38}$$

If the right-end support of the beam is free, one obtains the following boundary coefficient matrix:

$$[B_N] = \begin{array}{c} \begin{array}{cccc} 4N-3 & 4N-2 & 4N-1 & 4N \end{array} \\ \begin{bmatrix} -\sin \Omega & -\cos \Omega & \sinh \Omega & \cosh \Omega \\ -\cos \Omega & \sin \Omega & \cosh \Omega & \sinh \Omega \end{bmatrix} \end{array} \begin{array}{c} \\ q-1 \\ q \end{array}. \tag{39}$$

In Eq. (39), q denotes the total number of equations for the integration constants given by

$$q = 4N. \tag{40}$$

From the above derivations one sees that one may obtain four equations from each intermediate station at which a pinned support, external force, lumped mass, rotary inertia, linear spring or rotational spring is located and two equations from either left-end or right-end station of the beam. Therefore, the total number of equations for the integration constants is $q = 4N$.

4. Determination of natural frequencies, mode shapes and frequency–response curves of the beam

4.1. Determination of natural frequencies and mode shapes of the beam

The integration constants relating to the left-end support and those relating to the right-end support of the beam are defined by Eqs. (27) and (36), respectively, while those relating to the intermediate stations are determined by Eqs. (13), (17) and/or Eq. (21) depending upon concentrated element (such as lumped mass, rotary inertia, linear spring and/or rotational spring), pinned support and/or applied force being located there. The associated coefficient matrices are given by $[B_0]$ (cf. Eqs. (26), (29) or (30)), $[B_u]$, $[B_r]$, $[B_v]$ (cf. Eqs. (A.1)–(A.3) of the appendix), and $[B_N]$ (cf. Eqs. (35), (38) or (39)). From the last equations concerned one may see that the identification number for each element of the last four coefficient matrices is shown on the top side and right-hand side of each matrix. Therefore, using the numerical assembly technique as done by the conventional finite element method one may obtain a matrix equation for all integration constants of the entire beam

$$[\bar{B}]\{\bar{C}\} = \{\bar{D}\}. \tag{41}$$

For the case of free vibrations, the applied force amplitude $\bar{F}$ is zero and Eq. (41) reduces to

$$[\bar{B}]\{\bar{C}\} = 0. \tag{42}$$

In such a case, one must set $\omega_e = \omega$ with ω denoting natural frequency of the vibrating system.

Non-trivial solution of Eq. (42) requires that

$$|\bar{B}| = 0, \tag{43}$$

which is the frequency equation for the present problem.

In this paper, the incremental search method [15] is used to find the dimensionless frequency parameters, Ω_k ($k = 1,2,\ldots$). For each dimensionless frequency parameter Ω_k, one may obtain the corresponding integration constants from Eq. (42) and the substitution of last integration constants into displacement functions of the associated beam segments will determine the corresponding mode shape of the beam, $Y^{(k)}(\xi)$.

4.2. Determination of forced vibration response of the beam

For the case of forced vibrations, from Eq. (41) one has

$$\{\bar{C}\} = [\bar{B}]^{-1}\{\bar{D}\}. \tag{44}$$

Thus, if the exciting frequency ω_e (or the associated dimensionless frequency parameter Ω) of the harmonic forces is given, then one may obtain the corresponding integration constants from Eq. (44). The substitution of last integration constants into the displacement functions of associated beam segments will determine the corresponding vibration amplitude function of the beam, $Y(\xi)$.

5. Numerical results and discussions

Before the vibration analysis of a multi-span uniform beam carrying various concentrated elements is performed, the reliability of the theory and computer program developed for this paper are confirmed by comparing the present results with those obtained from existing literature or conventional finite element method (FEM). Unless otherwise mentioned, all numerical results of this paper are obtained based on a uniform Euler–Bernoulli beam with the following given data: Young's modulus $E = 2.069 \times 10^{11}\,\mathrm{N/m^2}$, diameter $d = 0.05\,\mathrm{m}$, moment of inertia of cross-sectional area $I = 3.06796 \times 10^{-7}\,\mathrm{m^4}$, mass per unit length $\bar{m} = 15.3879\,\mathrm{kg/m}$, and total length $L = 1\,\mathrm{m}$. In FEM, the two-node beam elements are used and each continuous beam is subdivided into 40 beam elements. Since each node has two degrees of freedom (dofs), the total dof for the entire beam is $2(40+1) = 82$.

5.1. Reliability of the developed computer program

The beam studied in this Example 1 is shown in Fig. 2. It is a uniform pinned–pinned beam carrying three point masses with rotary inertias at three locations, two linear springs and two rotational springs at the other two locations, and having two intermediate pinned supports. The given data for the three point masses and three rotary inertias are: $m_1^* = m_1/(\bar{m}L) = 0.3$, $m_5^* = 0.6$, $m_7^* = 0.9$ and $J_1^* = J_1/(\bar{m}L^3) = 0.001$, $J_5^* = 0.002$, $J_7^* = 0.003$ located at $\xi_1 = x_1/L = 0.1$, $\xi_5 = 0.6$ and $\xi_7 = 0.8$, respectively; those for the two linear springs and

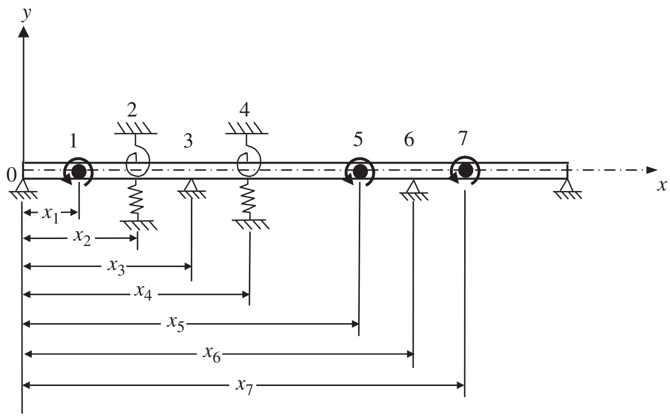

Fig. 2. Sketch for a pinned–pinned beam carrying three point masses, three rotary inertias, two linear springs, two rotational springs and with two intermediate pinned supports.

Table 1
The lowest five dimensionless natural frequency parameters of a uniform beam carrying three point masses, three rotary inertias, two linear springs, two rotational springs, and with two intermediate pinned supports

Boundary conditions	Methods	Dimensionless frequency parameters				
		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
P–P	Present	6.613083	8.214078	9.235993	11.506641	13.353669
	FEM	6.605338	8.204520	9.225176	11.493720	13.338130
C–F	Present	4.089879	7.633916	10.002664	10.948062	11.986887
	FEM	4.089880	7.633918	10.002671	10.948070	11.986894

two rotational springs are: $K^*_{T2} = K_{T2}L^3/(EI) = 10$, $K^*_{T4} = 20$ and $K^*_{R2} = K_{R2}L/(EI) = 3$, $K^*_{R4} = 4$ located at $\xi_2 = 0.2$ and $\xi_4 = 0.4$, respectively; and the two intermediate pinned supports are located at $\xi_3 = 0.3$ and $\xi_6 = 0.7$. Two types of boundary conditions (P–P and C–F) are studied. Where P, C and F represent the abbreviations of pinned, clamped and free, respectively. The lowest five natural frequency parameters of the beam are shown in Table 1. It is seen that the results of the present paper are in excellent agreement with those of FEM. Figs. 3(a and b) show the lowest five mode shapes of the uniform beam with P–P and C–F boundary conditions, respectively. In which, the 1st, 2nd, 3rd, 4th and 5th mode shapes are represented by the curves ———, ··········, — - —, — — — and —·—·—, respectively.

The beam studied in Example 2 is a cantilever beam as shown in Fig. 4. It is simply supported at $\xi_1 = 0.5$ and subjected to the action of a harmonic concentrated force $\bar{F}_2 \mathrm{e}^{\mathrm{j}\omega_e t}$ at free end (located at $\xi_2 = 1.0$). The dimensionless frequency parameter and magnitude of the applied force are $\Omega = \sqrt{5}$ and $\bar{F}_2 = 1$, respectively. The dimensionless vibration amplitudes $(\bar{Y}(\xi)_{\max} = Y(\xi)_{\max}/(\bar{F}L^3/EI))$at different locations of the beam are given in Table 2. It is seen that the current numerical results are also in excellent agreement with those of Ref. [14].

5.2. *Forced vibration responses of a multiple-span beam carrying a number of various concentrated elements and subjected to a harmonic force*

In this subsection, the force vibration responses of the uniform beam shown in Fig. 2 subjected to the action of a harmonic concentrated force $F_v(t) = \bar{F}_v \mathrm{e}^{\mathrm{j}\omega_e t}$ located at various positions along the beam length are studied (cf. Fig. 5).

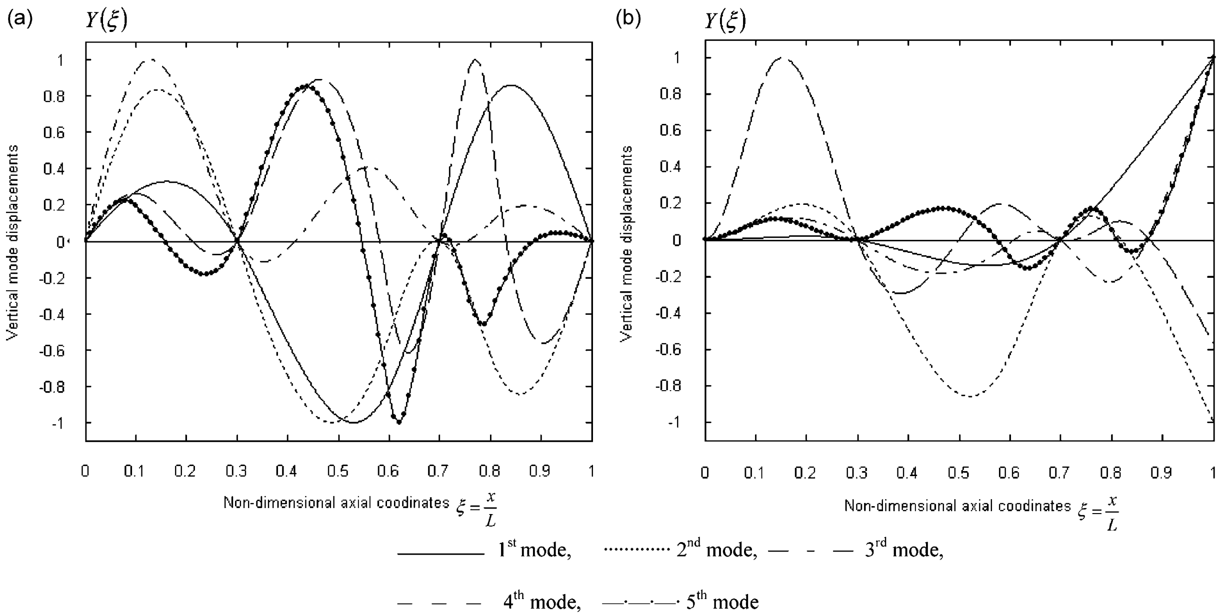

Fig. 3. The lowest five mode shapes of a uniform beam carrying three point masses, three rotary inertias, two linear springs, two rotational springs, and with two intermediate pinned supports: (a) pinned–pinned (P–P) beam and (b) cantilever (C–F) beam.

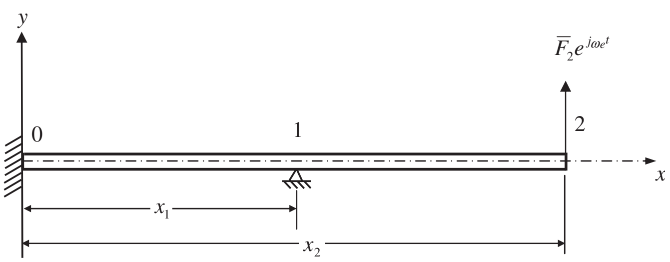

Fig. 4. Sketch for a cantilever beam with an intermediate pinned support and subjected to the action of a harmonic concentrated force $\bar{F}_2\,\mathrm{e}^{\mathrm{j}\omega_e t}$ at free end.

Table 2
The dimensionless vibration amplitudes at different locations of the cantilever beam with an intermediate pined-support and subjected to a harmonic concentrated force at free end (cf. Fig. 4)

Dimensionless coordinates, $\xi = x/L$	Dimensionless amplitudes $\bar{Y}(\xi)_{\max}$	
	Present	Ref. [14]
0.0	0.000000	0.000000
0.1	−0.001380	−0.001380
0.2	−0.004136	−0.004138
0.3	−0.006197	−0.006198
0.4	−0.005501	−0.005503
0.5	0.000000	0.000000
0.6	0.011747	0.011742
0.7	0.028814	0.028804
0.8	0.049712	0.049702
0.9	0.073026	0.073011
1.0	0.097467	0.097442

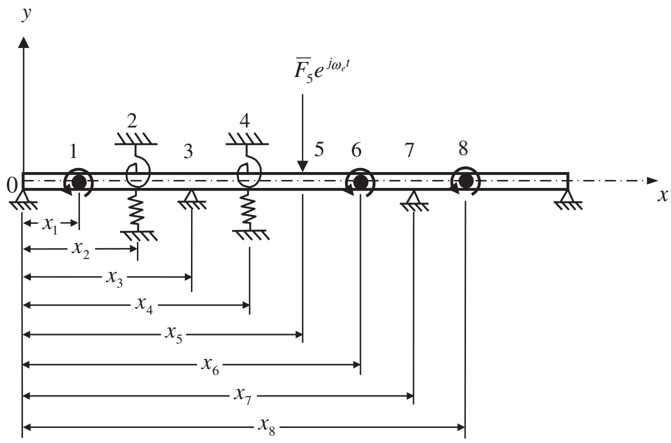

Fig. 5. Sketch for the pinned–pinned beam shown in Fig. 2 subjected to the action of a harmonic force $F_5(t) = \bar{F}_2 \, \mathrm{e}^{\mathrm{j}\omega_e t}$.

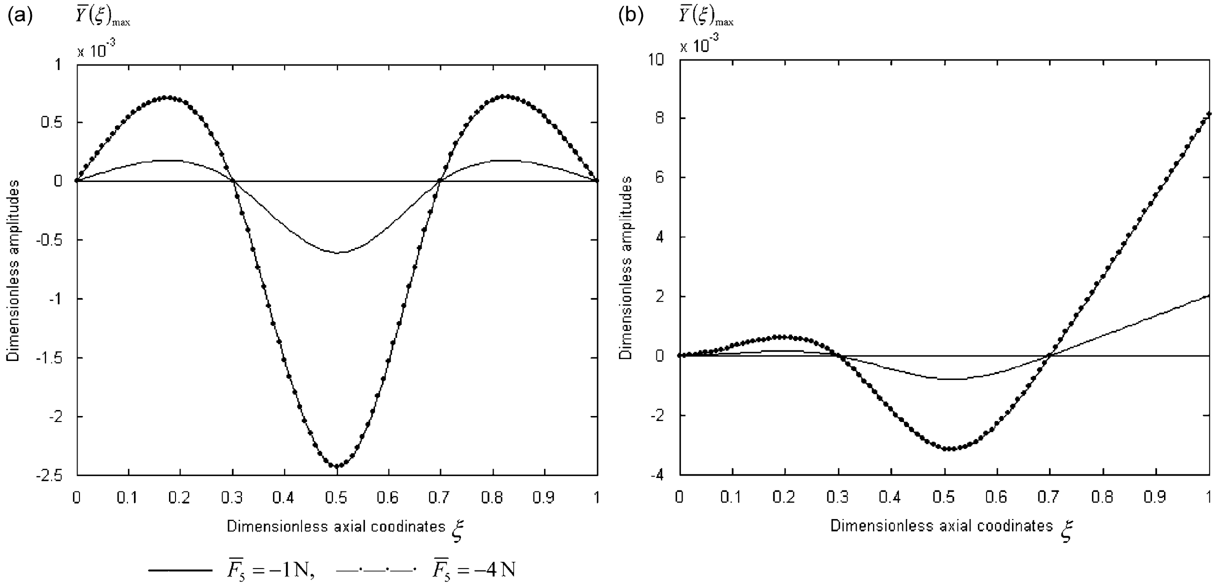

Fig. 6. The dimensionless vibration amplitudes $\bar{Y}(\xi)_{\max}$ of the beam carrying five concentrated elements with two intermediate pinned supports and subjected to a harmonic force at $\xi_5 = 0.5$ (cf. Fig. 5). (a) Pinned–pinned (P–P) beam and (b) cantilever (C–F) beam.

5.2.1. *Forced vibration response amplitudes of the entire beam*

For a harmonic force with dimensionless frequency parameter $\Omega = \sqrt{5}$ and force amplitude $\bar{F}_5 = -1\,\mathrm{N}$ or $\bar{F}_5 = -4\,\mathrm{N}$ located at $\xi_5 = 0.5$, Figs. 6(a) and (b) show the dimensionless vibration amplitudes of the beam with P–P and C–F boundary conditions, respectively. In which, the dimensionless vibration amplitudes $(\bar{Y}(\xi)_{\max} = Y(\xi)_{\max}/(\bar{F}_5 L^3/EI) = Y(\xi)_{\max}/(1 \times 1^3/(2.069 \times 10^{11} \times 3.06796 \times 10^{-7})))$ due to $\bar{F}_5 = -1\,\mathrm{N}$ and $\bar{F}_5 = -4\,\mathrm{N}$ are represented by the curves ——— and —·—·—·, respectively.

5.2.2. *Frequency–response curves for several points*

The present example is the same as the last one, but the force amplitude is $\bar{F}_v = -1\,\mathrm{N}$ and the dimensionless frequency parameter Ω is variable. For each value of Ω (from 0 to 14.0 with interval $\Delta\Omega = 0.001$), one may obtain the integration constants from Eq. (44). Then, by using Eq. (6), we compute the vibration amplitudes

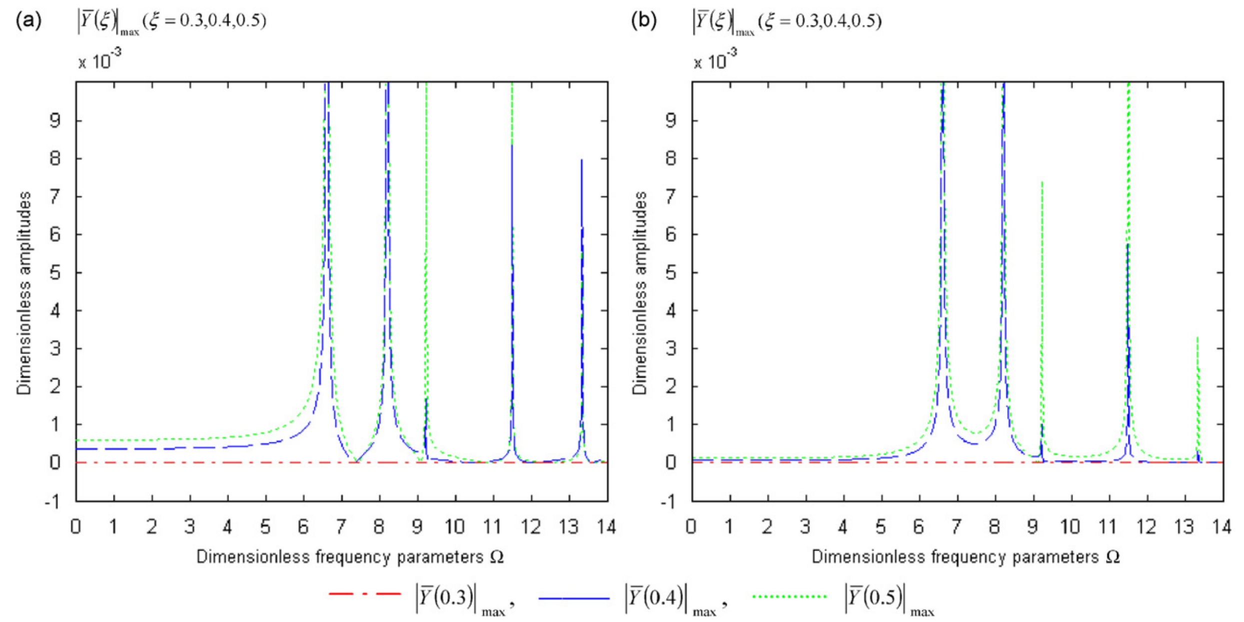

Fig. 7. The relationship for dimensionless frequency parameters (Ω) versus dimensionless amplitudes ($|\bar{Y}(\xi)|_{\max}$) for the points located at $\xi = 0.3$, 0.4 and 0.5 of the pinned–pinned (P–P) beam subjected to a harmonic force with amplitude $\bar{F}_v = -1\,\mathrm{N}$ applied at: (a) $\xi_5 = 0.5$ and (b) $\xi_9 = 0.9$ (cf. Fig. 5).

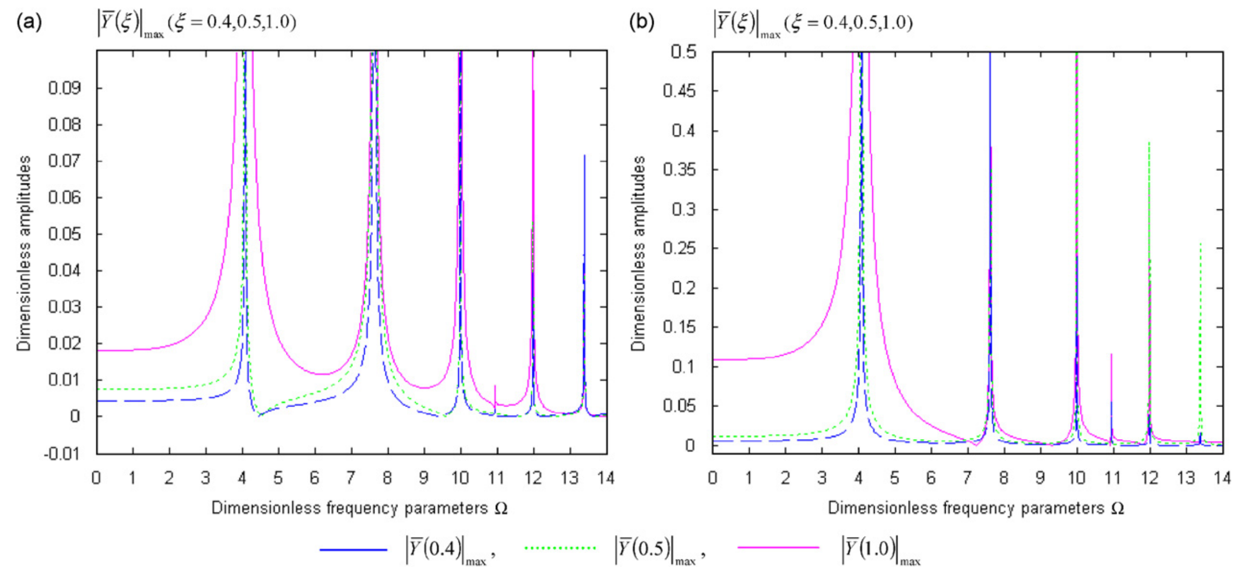

Fig. 8. The relationship for dimensionless frequency parameters (Ω) versus dimensionless amplitudes ($|\bar{Y}(\xi)|_{\max}$) for the points located at $\xi = 0.4$, 0.5 and 1.0 of the cantilever (C–F) beam subjected to a harmonic force with amplitude $\bar{F}_v = -1\,\mathrm{N}$ applied at (a) $\xi_5 = 0.5$ and (b) $\xi_9 = 0.9$ (cf. Fig. 5).

when it is placed in different locations on the beam. Fig. 7(a) shows the relationship between the dimensionless frequency parameters (Ω) and the dimensionless response amplitudes (($|\bar{Y}(\xi)|_{\max} = |Y(\xi)|_{\max}/(\bar{F}L^3/EI)$)) for the points located at $\xi = 0.3$, 0.4 and 0.5 of the P–P beam subjected to a harmonic force with amplitude $\bar{F}_v = -1\,\mathrm{N}$ applied at $\xi_5 = 0.5$, where the horizontal axis is the dimensionless frequency parameter (Ω) and the vertical axis is the dimensionless vibration amplitude ($|\bar{Y}(\xi)|_{\max}$). The red (— - —), blue (— — —) and green (⋯⋯⋯) curves are for points located at locations $\xi = 0.3$, 0.4 and 0.5, respectively. The vibration amplitudes for the point located at $\xi = 0.3$ are equal to 0 because it is on the pinned support of the beam. The curves for the point located at either $\xi = 0.4$ or 0.5 have peaks when the dimensionless frequency parameter $\Omega \approx 6.6$, 8.2, 9.2, 11.5 and 13.3. This is because when the dimensionless frequency parameter Ω is near any of the natural frequencies of the P–P beam, as shown in line 1 of Table 1, resonance appears. The frequency–response curves shown in Fig. 7(b) are similar to those shown in Fig. 7(a) (webfigure), the only difference is the harmonic force $\bar{F}_v = -1$ N being applied at $\xi_9 = 0.9$ (rather than at $\xi_5 = 0.5$ for Fig. 7(a)). All conditions for Fig. 8 (web figure) are the same as those for Fig. 7 except that: (i) the beam is with clamped–free (C–F) boundary conditions; (ii) the frequency–response curves for the points located at $\xi = 0.4$, 0.5 and 1.0 are determined. In Fig. 8, the frequency responses for the points located at $\xi = 0.4$, 0.5 and 1.0 are represented by the blue (— — —), green (⋯⋯⋯) and purple (————) curves, respectively. It is the same as Fig. 7 that Fig. 8(a) is for the case of the harmonic force $\bar{F}_v = -1$ N being applied at $\xi_5 = 0.5$ and Fig. 8(b) is at $\xi_9 = 0.9$. It is seen that each frequency–response curve has a peak when $\Omega \approx 4.1$, 7.6, 10.0, 10.9 and 12.0. Likewise, this is because when the dimensionless frequency parameter Ω is near any of the natural frequencies of the C–F beam, as shown in line 3 of Table 1, resonance appears.

6. Conclusions

1. Using the numerical assembly method (NAM), one can obtain the "exact" solutions for the natural frequencies and mode shapes of a uniform multi-span beam carrying a number of various concentrated elements under different boundary conditions.
2. Using the NAM, one can also determine the "exact" vibration amplitude of the entire beam when it is subjected to a harmonious force with a specified exciting frequency.
3. For a beam subjected to a harmonic force, one can determine the frequency–response curve for any point of the beam using NAM. Because a peak will appear in each curve when the exciting frequency of the harmonic force is near any of natural frequencies of the beam, one can determine natural frequencies of the beam based on the peaks of any frequency–response curve.

Appendix

$$[B_u] = \begin{array}{c} \begin{array}{cccccccc} 4u-3 & 4u-2 & 4u-1 & 4u & 4u+1 & 4u+2 & 4u+3 & 4u+4 \end{array} \\ \left[\begin{array}{cccccccc} s\theta_u & c\theta_u & sh\theta_u & ch\theta_u & -s\theta_u & -c\theta_u & -sh\theta_u & -ch\theta_u \\ c\theta_u & -s\theta_u & ch\theta_u & sh\theta_u & -c\theta_u & s\theta_u & -ch\theta_u & -sh\theta_u \\ -\Omega s\theta_u + \alpha_u c\theta_u & -\Omega c\theta_u - \alpha_u s\theta_u & \Omega sh\theta_u + \alpha_u ch\theta_u & \Omega ch\theta_u + \alpha_u sh\theta_u & \Omega s\theta_u & \Omega c\theta_u & -\Omega sh\theta_u & -\Omega ch\theta_u \\ \delta_u s\theta_u - \Omega^3 c\theta_u & \delta_u c\theta_u + \Omega^3 s\theta_u & \delta_u sh\theta_u + \Omega^3 ch\theta_u & \delta_u ch\theta_u + \Omega^3 sh\theta_u & \Omega^3 c\theta_u & -\Omega^3 s\theta_u & -\Omega^3 ch\theta_u & -\Omega^3 sh\theta_u \end{array}\right] \end{array} \begin{array}{c} \\ 4u-1 \\ 4u \\ 4u+1 \\ 4u+2 \end{array}, \tag{A.1}$$

where $\theta_u = \Omega\xi_u$, $s\theta_u = \sin\Omega\xi_u$, $c\theta_u = \cos\Omega\xi_u$, $sh\,\theta_u = \sinh\Omega\xi_u$, $ch\,\theta_u = \cosh\Omega\xi_u$, $\alpha_u = -\Omega^4 J_u^* + K_{Ru}^*$ and $\delta_u = \Omega^4 m_u^* - K_{Tu}^*$.

$$[B_r] = \begin{array}{c} \begin{array}{cccccccc} 4r-3 & 4r-2 & 4r-1 & 4r & 4r+1 & 4r+2 & 4r+3 & 4r+4 \end{array} \\ \left[\begin{array}{cccccccc} s\theta_r & c\theta_r & sh\theta_r & ch\theta_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s\theta_r & c\theta_r & sh\theta_r & ch\theta_r \\ c\theta_r & -s\theta_r & ch\theta_r & sh\theta_r & -c\theta_r & s\theta_r & -ch\theta_r & -sh\theta_r \\ -s\theta_r & -c\theta_r & sh\theta_r & ch\theta_r & s\theta_r & c\theta_r & -sh\theta_r & -ch\theta_r \end{array}\right] \end{array} \begin{array}{c} \\ 4r-1 \\ 4r \\ 4r+1 \\ 4r+2 \end{array}, \tag{A.2}$$

where $\theta_r = \Omega\xi_r$, $s\theta_r = \sin\Omega\xi_r$, $c\theta_r = \cos\Omega\xi_r$, $sh\,\theta_r = \sinh\Omega\xi_r$ and $ch\,\theta_r = \cosh\Omega\xi_r$.

$$
[B_v] = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccccccc|c}
4v-3 & 4v-2 & 4v-1 & 4v & 4v+1 & 4v+2 & 4v+3 & 4v+4 & \\
\hline
s\theta_v & c\theta_v & sh\theta_v & ch\theta_v & -s\theta_v & -c\theta_v & -sh\theta_v & -ch\theta_v & 4v-1 \\
c\theta_v & -s\theta_v & ch\theta_v & sh\theta_v & -c\theta_v & s\theta_v & -ch\theta_v & -sh\theta_v & 4v \\
-s\theta_v & -c\theta_v & sh\theta_v & ch\theta_v & s\theta_v & c\theta_v & -sh\theta_v & -ch\theta_v & 4v+1 \\
-\Omega^3 c\theta_v & \Omega^3 s\theta_v & \Omega^3 ch\theta_v & \Omega^3 sh\theta_v & \Omega^3 c\theta_v & -\Omega^3 s\theta_v & -\Omega^3 ch\theta_v & -\Omega^3 sh\theta_v & 4v+2
\end{array}, \tag{A.3}
$$

where $\theta_v = \Omega\xi_v$, $s\theta_v = \sin\Omega\xi_v$, $c\theta_v = \cos\Omega\xi_v$, $sh\,\theta_v = \sinh\Omega\xi_v$ and $ch\,\theta_v = \cosh\Omega\xi_v$.

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